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On multilinear oscillatory singular integrals with rough kernels ☆

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Abstract

In this paper, for the multilinear oscillatory singular integral operators T_A defined by

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A; x, y) f(y)\, dy, \quad n \geqslant 2,$$

where $P(x, y)$ is a nontrivial and real-valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$, $\Omega(x)$ is homogeneous of degree zero on $\mathbb{R}^n$, $A(x)$ has derivatives of order m in $\dot{\Lambda}_\beta$ $(0 < \beta < 1)$, $R_{m+1}(A; x, y)$ denotes the $(m+1)$th remainder of the Taylor series of A at x expended about y, the author proves that if $\Omega \in L^q(S^{n-1})$ for some $q > 1/(1-\beta)$, then for any $p \in (1, \infty)$, T_A is bounded on $L^p(\mathbb{R}^n)$. Meanwhile, the weighted L^p-boundedness of T_A is also given.

1. Introduction

As is well known, oscillatory singular integral operators with polynomial phase are very useful in the study of Hilbert transforms along curves, singular integrals supported on lower-dimensional varieties and singular Radon transforms. There has been significant progress in the study of this type of operators since Ricci and Stein [23] gave the prototypical work in this area (see [3,9,10,15,16,18,22,23] et al. and references theirein).

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In this paper, we will consider the following multilinear operator related to oscillatory singular integral defined by

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y)\,dy, \quad n \geqslant 2,$$

where $P(x,y)$ is a real polynomial on $\mathbb{R}^n$, Ω is homogeneous of degree zero on $\mathbb{R}^n$ and $\int_{S^{n-1}} \Omega(x')\,dx' = 0$, S^{n-1} denotes the unit sphere of $\mathbb{R}^n$, $A(x)$ has derivatives of order m in $\mathbb{R}^n$, $R_{m+1}(A;x,y)$ is the $(m+1)$th $(m \geqslant 1)$ order remainder of the Taylor series of A expended at x about y, precisely,

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\gamma| \leqslant m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma.$$

For operators of this type, there have been many interesting works (see [5–7,14] et al.). Here, we consider the case that $D^\gamma A \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ $(|\gamma| \leqslant m)$, where $\dot{\Lambda}_\beta$ denotes the Lipschitz space defined by

$$\dot{\Lambda}_\beta(\mathbb{R}^n) = \left\{ f\colon\ \|f\|_{\dot{\Lambda}_\beta} = \sup_{x,\,h \in \mathbb{R}^n,\, h \neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty \right\},$$

where $\Delta_h^1 f(x) = f(x+h) - f(x)$, $\Delta_h^{k+1} f(x) = \Delta_h^1(\Delta_h^{k-1} f)(x)$. It is easy to see that if $0 < \beta < 1$, $f(x) \in \dot{\Lambda}_\beta$, then

$$\big|f(x) - f(y)\big| \leqslant |x-y|^\beta \|f\|_{\dot{\Lambda}_\beta}, \quad \forall x, y \in \mathbb{R}^n. \tag{1.1}$$

For the corresponding multilinear operator related to singular integral defined by

$$\bar{T}_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y)\,dy,$$

Chen [4] showed that if $\Omega \in \mathrm{Lip}_1(S^{n-1})$, then for $1/r = 1/p - \beta/n$,

$$\|\bar{T}_A f\|_r \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_p.$$

Recently, Lu et al. [17] improved the above result to the case $\Omega \in L^q(S^{n-1})$ for some $q \geqslant n/(n-\beta)$. These results indicates that for $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m)$, $\bar{T}_A$ has the same mapping properties on the Lebesgue spaces as those of the fractional integral operator $\bar{T}$ defined by (see [11,19,20] et al.)

$$\bar{T} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\beta}} f(y)\,dy.$$

It is naturally led to the question whether T_A has the same mapping properties on L^p as those of the fractional oscillatory integral operator.

For the fractional oscillatory singular integral operator with smoothness kernel, Ricci and Stein [23] showed the following result.

Theorem A (cf. [23]). *For each $d \geqslant 2$, there exists $a_d > 0$, such that whenever*

(i) *$P(x, y)$ is a real polynomial of total degree $\leqslant d$, which is nontrivial in the sense that it cannot be written as $P_0(x) + P_1(y)$, and*
(ii) *$K(x, y)$ is a function which satisfies $|K(x, y)| \leqslant C|x - y|^{-n+\beta}$, $|\nabla K(x, y)| \leqslant C|x - y|^{-n+\beta-1}$,*

then the operator T defined by

$$Tf(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} K(x, y) f(y)\, dy$$

is bounded on $L^p(\mathbb{R}^n)$, where $0 < \beta < a_d(1/2 - |1/p - 1/2|)$, and the bound of the operator do depend on the polynomial $P(x, y)$.

In 1996, Y. Ding [9] improved the above result as follows.

Theorem B (cf. [9]). *Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and belongs to $L^q(S^{n-1})$ for some $q > 1$, $b(r) \in BV(\mathbb{R}_+)$, $P(x, y) = \sum_{|\xi| \leqslant k,\, |\eta| \leqslant l} a_{\xi\eta} x^\xi y^\eta$ is a nontrivial polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. Then for the fractional oscillatory singular integral operator*

$$Tf(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x - y)}{|x - y|^{n-\beta}} b\bigl(|x - y|\bigr) f(y)\, dy,$$

(i) *if $0 < \beta < \min\{(l + k)/2k, (l + k)/2l\}$ and $q > 1/(1 - \beta)$, then T is bounded on $L^2(\mathbb{R}^n)$;*
(ii) *if $1 < p < \infty$ $(p \neq 2)$, $0 < \beta < \min\{(l + k)/2k, (l + k)/2l\}\{1/2 - |1/p - 1/2|\}$ and $q > 1/(1 - \beta)$, then T is bounded on $L^p(\mathbb{R}^n)$.*

Here the bound of T depend on the value of $\sum_{|\xi|=k,\, |\eta|=l} |a_{\xi\eta}|$, but not on the other coefficients of $P(x, y)$.

The aim of this paper is to show that T_A enjoys some properties, which are parallel to those of the fractional oscillatory integral operator T, and to give a positive answer to the above question. Our main result can be stated as follows.

Theorem 1. *Suppose that Ω, $R_{m+1}(A; x, y)$ is as above, $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m)$, $P(x, y) = \sum_{|\xi| \leqslant k,\, |\eta| \leqslant l} a_{\xi\eta} x^\xi y^\eta$ is a nontrivial polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. Then for $\Omega \in L^q(S^{n-1})$,*

(i) *if $0 < \beta < \min\{(l + k)/2k, (l + k)/2l\}$ and $q > 1/(1 - \beta)$, then*

$$\|T_A f\|_2 \leqslant C(n, m, \deg P) \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_2;$$

(ii) *if* $1 < p < \infty$ $(p \neq 2)$, $0 < \beta < \min\{(l+k)/2k, (l+k)/2l\}\{1/2 - |1/p - 1/2|\}$ *and* $q > 1/(1-\beta)$, *then*

$$\|T_A f\|_p \leqslant C(n, m, \deg P) \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_p.$$

Remark 1. It is worth pointing out that the bound of the fractional oscillatory operators in Theorems A and B do depend on the coefficients of $P(x, y)$, but the bound of T_A in our theorem is independent of the coefficients of $P(x, y)$.

In addition, we shall also establish the weighted analog of Theorem 1. Suppose that ω is nonnegative and locally integrable function, recall that $\omega(\cdot) \in A_p$, $1 < p < \infty$, if for some constant $C > 0$,

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x)\, dx\right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)}\, dx\right)^{p-1} \leqslant C < \infty,$$

where Q are the cubes whose sides are parallel to the coordinate axes.

The following properties can be found in [13]:

(a) If $1 \leqslant p_1 \leqslant p_2 < \infty$, then $A_{p_1} \subseteq A_{p_2}$;
(b) If $\omega(x) \in A_p$, then there exists $\varepsilon > 0$ such that $\omega(x)^{\varepsilon+1} \in A_p$;
(c) If $\omega(x) \in A_p$, then for any $\delta \in [0, 1)$, $\omega(x)^{1+\delta} \in A_p$.

For a weight ω on $\mathbb{R}^n$, we write $\|f\|_{p,\omega} = \|f\omega^{1/p}\|_p$ and $\omega(E) = \int_E \omega(x)\, dx$. The weighted version of Theorem 1 is as follows.

Theorem 2. *Let* Ω, $R_{m+1}(A; x, y)$ *be as above,* $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m)$, $1 < p < \infty$, $0 < \sigma = 2\varepsilon/p(1+\varepsilon)$ *and* $0 < \beta < \sigma \min\{(l+k)/2k,\ (l+k)/2l\}$. *If* p, q, ε, *and the weight functions* $\omega(x)$ *satisfy one of the following conditions, then for* $\Omega \in L^q(S^{n-1})$,

$$\|T_A f\|_{p,\omega} \leqslant C(n, m, \deg P) \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_{p,\omega}.$$

(i) $2 < p < \infty$, $q' \leqslant p$, $\omega(x) \in A_{p/q'}$, $\varepsilon \in (0, 1)$ *such that* $\omega(x)^{1+\varepsilon} \in A_{p/q'}$.
(ii) $1 < p < 2$, $p \leqslant q$, $\omega(x)^{-1/(p-1)} \in A_{p'/q'}$. *For* μ *satisfying* $(\omega(x)^{-1/(p-1)})^{(1+\mu)} \in A_{p'/q'}$, *choose*

$$0 < \varepsilon < \min\bigl\{(p-1)/2, \mu(p-1)/(1+\mu)(2-p)\bigr\}$$

such that $(\omega(x)^{-1/(p-1)})^{(1+\mu)(1+\varepsilon)} \in A_{p'/q'}$.
(iii) $p = 2$, $q > 1/(1-\beta)$, $\omega(x)^{q'} \in A_2$, $\varepsilon > 0$ *such that* $(\omega(x)^{q'})^{1+\varepsilon} \in A_2$.

Remark 2. It is sure from (a) and (b) that ε's and μ's in Theorem 2 can be chosen.

This paper is organized as follows. In Section 2, we will give some preliminary lemmas. Next we will prove Theorem 1 in Section 3. Finally, the proof of Theorem 2 will be given in Section 4. We would remark that our some ideas in the proofs of our theorems are taken from [9,10,14,23]. Throughout the rest of this paper, we always use the letter C to denote positive constants that may vary at each occurrence, but are independent of the essential variables.

2. Some lemmas

In this section, we give some preliminary lemmas.

Lemma 1 (cf. [8]). *Let $A(x)$ be a function on $\mathbb{R}^n$ with mth order derivatives in $L^s_{\mathrm{loc}}(\mathbb{R}^n)$ for some $s > n$. Then*

$$\left|R_m(A; x, y)\right| \leqslant C_{m,n}|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{|Q_x^y|} \int_{Q_x^y} \left|D^\gamma A(z)\right|^s dz\right)^{1/s},$$

where Q_x^y is the cube centered at x with diameter $5\sqrt{n}\,|x-y|$.

Lemma 2 (cf. [21]). *Let $0 < \beta < 1$, $1 \leqslant q < \infty$, we have*

$$\|f\|_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{Q} \int_Q \left|f(x) - m_Q(f)\right|^q dx\right)^{1/q},$$

where $m_Q(f) = 1/|Q| \int_Q f(x)\,dx$. For $q = \infty$, the formula should be interpreted appropriately.

Lemma 3 (cf. [21]). *Let $Q^* \subset Q$, $g \in \dot{\Lambda}_\beta$ $(0 < \beta < 1)$. Then*

$$\left|m_{Q^*}(g) - m_Q(g)\right| \leqslant C|Q|^{\beta/n}\|g\|_{\dot{\Lambda}_\beta}.$$

Lemma 4. *Let Q be a cube centered at x with diameter r. If $D^\gamma A \in \dot{\Lambda}_\beta$ $(0 < \beta < 1$, $|\gamma| = m)$, then for $|x-y| < r$,*

$$\left|R_{m+1}(A; x, y)\right| \leqslant Cr^\beta|x-y|^m \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta}.$$

Proof. Set

$$A_Q(y) = A(y) - \sum_{|\gamma|=m} \frac{1}{\gamma!} m_Q(D^\gamma A) y^\gamma.$$

It is easy to verify that $R_{m+1}(A; x, y) = R_{m+1}(A_Q; x, y)$. By Lemma 1, we get

$$\begin{aligned}
\left|R_{m+1}(A_Q; x, y)\right| &\leqslant \left|R_m(A_Q; x, y)\right| + C \sum_{|\gamma|=m} \left|D^\gamma A_Q(y)\right||x-y|^m \\
&\leqslant C|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{|Q_x^y|} \int_{Q_x^y} \left|D^\gamma A_Q(z)\right|^s dz\right)^{1/s} + C|x-y|^m \sum_{|\gamma|=m} \left|D^\gamma A_Q(y)\right|,
\end{aligned}$$

where Q_x^y is the cube centered at x with diameter $5\sqrt{n}\,|x-y|$. Notice that if $|x-y|<r$, then $Q_x^y \subset 5nQ$. By Lemmas 2 and 3, we have

$$\begin{aligned}
\left(\frac{1}{|Q_x^y|}\int\limits_{Q_x^y}\left|D^\gamma A_Q(z)\right|^s dz\right)^{1/s} &= \left(\frac{1}{|Q_x^y|}\int\limits_{Q_x^y}\left|D^\gamma A(z)-m_Q(D^\gamma A)\right|^s dz\right)^{1/s}\\
&\leqslant \left(\frac{1}{|Q_x^y|}\int\limits_{Q_x^y}\left|D^\gamma A(z)-m_{Q_x^y}(D^\gamma A)\right|^s dz\right)^{1/s} + \left|m_{Q_x^y}(D^\gamma A)-m_{5nQ}(D^\gamma A)\right|\\
&\quad + \left|m_{5nQ}(D^\gamma A)-m_Q(D^\gamma A)\right|\\
&\leqslant C|Q|^{\beta/n}\|D^\gamma A\|_{\dot{\Lambda}_\beta} \leqslant Cr^\beta\|D^\gamma A\|_{\dot{\Lambda}_\beta},
\end{aligned}$$

and

$$\left|D^\gamma A_Q(y)\right| = \left|D^\gamma A(y)-m_Q(D^\gamma A)\right| \leqslant C|Q|^{\beta/n}\|D^\gamma A\|_{\dot{\Lambda}_\beta} \leqslant Cr^\beta\|D^\gamma\|_{\dot{\Lambda}_\beta}.$$

Hence

$$\left|R_{m+1}(A;x,y)\right| = \left|R_{m+1}(A_Q;x,y)\right| \leqslant Cr^\beta|x-y|^m\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta},$$

which completes the proof of Lemma 4. □

3. Proof of Theorem 1

By dilation-invariance, we may assume that $\sum_{|\xi|=k,\,|\eta|=l}|a_{\xi\eta}|=1$. Write

$$\begin{aligned}
T_A f(x) &= \int\limits_{|x-y|<1} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}R_{m+1}(A;x,y)f(y)\,dy\\
&\quad + \int\limits_{|x-y|\geqslant 1} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}R_{m+1}(A;x,y)f(y)\,dy\\
&:= T_A^0 f(x) + T_A^\infty f(x).
\end{aligned}\tag{3.1}$$

At first, we estimate $\|T_A^0 f\|_p$, $1<p<\infty$. By Lemma 4, we have

$$\begin{aligned}
\left|T_A^0 f(x)\right| &\leqslant \int\limits_{|x-y|<1}\frac{|\Omega(x-y)|}{|x-y|^{n+m}}\left|R_{m+1}(A;x,y)\right|\left|f(y)\right|dy\\
&= \sum_{j=0}^\infty\int\limits_{2^{-j-1}\leqslant|x-y|<2^{-j}}\frac{|\Omega(x-y)|}{|x-y|^{n+m}}\left|R_{m+1}(A;x,y)\right|\left|f(y)\right|dy\\
&\leqslant C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\sum_{j=0}^\infty 2^{-j\beta}\int\limits_{2^{-j-1}\leqslant|x-y|<2^{-j}}\frac{|\Omega(x-y)|}{|x-y|^n}\left|f(y)\right|dy
\end{aligned}$$

$$\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} \sum_{j=0}^{\infty} 2^{-j\beta} 2^{jn} \int\limits_{|x-y|<2^{-j}} \bigl|\Omega(x-y)\bigr|\bigl|f(y)\bigr|\,dy$$

$$\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} \sum_{j=0}^{\infty} 2^{-j\beta} M_\Omega f(x)$$

$$\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} M_\Omega f(x),$$

where M_Ω is the maximal operator with rough kernel defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{|x-y|<r} \bigl|\Omega(x-y)\bigr|\bigl|f(y)\bigr|\,dy.$$

Since $\Omega \in L^q(S^{n-1})$ $(q>1)$, it follows from [2] that for any $1<p<\infty$,

$$\|M_\Omega f\|_p \leqslant C\|f\|_p.$$

Thus

$$\begin{aligned}\bigl\|T_A^0 f\bigr\|_p &\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} \|M_\Omega f\|_p \\ &\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} \|f\|_p, \quad 1<p<\infty, \end{aligned} \tag{3.2}$$

where C is independent of the coefficients of $P(x,y)$. It remains to show that

$$\bigl\|T_A^\infty f\bigr\|_p \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} \|f\|_p. \tag{3.3}$$

By letting

$$T_A^j f(x) = \int\limits_{2^j \leqslant |x-y|<2^{j+1}} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A;x,y) f(y)\,dy,$$

we have

$$T_A^\infty f(x) = \sum_{j=0}^{\infty} T_A^j f(x).$$

Obviously, in order to obtain (3.3), we only need to prove that there is a constant $\theta>0$ such that for every $1 \leqslant j<\infty$,

$$\bigl\|T_A^j f\bigr\|_p \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} 2^{-j\theta} \|f\|_p, \tag{3.4}$$

where C is independent of f and j.

We turn our attention to the operator

$$\tilde{T}_A^j f(x) = 2^{j\beta} \int\limits_{1\leqslant|x-y|<2} e^{iP(2^j x, 2^j y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A; x, y) f(y)\, dy.$$

It is easy to check up that the proof of (3.4) can be reduced to showing that

$$\left\| \tilde{T}_A^j f \right\|_p \leqslant C 2^{-\theta j} \sum_{|\gamma|=m} \| D^\gamma A \|_{\dot{\Lambda}_\beta} \| f \|_p. \tag{3.5}$$

For fixed $j \in \mathbb{N}$, we now prove (3.5). Write $\mathbb{R}^n = \bigcup_d Q_d$, where each Q_d is a cube with side length 1 and the cube have disjoint interiors. Set $f_d = f\chi_{Q_d}$. Since that the support of $\tilde{T}_A^j f_d$ is contained in a fixed multiple of Q_d, so the supports of various terms $\tilde{T}_A^j f_d$ have bounded overlaps. Thus

$$\left\| \tilde{T}_A^j f \right\|_p^p \leqslant C \sum_d \left\| \tilde{T}_A^j f_d \right\|_p^p. \tag{3.6}$$

For each fixed d, denote $\bar{Q}_d = 10n Q_d$. From [8] we can take $\varphi_d(x) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leqslant \varphi_d \leqslant 1$, φ_d is identically one on $4\sqrt{n}\, Q_d$ and vanishes outside of $6\sqrt{n}\, Q_d$, $\| D^\gamma \varphi_d \|_\infty \leqslant C |\bar{Q}_d|^{-|\gamma|/n}$ for all multi-index γ ($|\gamma| \leqslant m$).

Let x_0 be a point on the boundary of $8\sqrt{n}\, Q_d$. Denote

$$A_{Q_d}(y) = A(y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} m_{\bar{Q}_d}(D^\alpha A) y^\alpha,$$

$$A_{\varphi_d}(y) = R_m(A_{Q_d}; y, x_0)\varphi_d(y),$$

and for multi-index α,

$$\tilde{T}_j^\alpha h(x) = 2^{j\beta} \int\limits_{1\leqslant|x-y|<2} e^{iP(2^j x, 2^j y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} (x-y)^\alpha h(y)\, dy.$$

It is easy to deduce that (see [8, (25)])

$$\begin{aligned}
\tilde{T}_A^j f_d(x) &= \tilde{T}_{A_{\varphi_d}}^j f_d(x) \\
&= A_{\varphi_d}(x) \tilde{T}_j^0 f(x) - \sum_{0<|\alpha|<m} \frac{1}{\alpha!} \tilde{T}_j^\alpha (D^\alpha A_{\varphi_d} f_d)(x) \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \tilde{T}_j^\alpha (D^\alpha A_{\varphi_d} f_d)(x) \\
&:= G + H + J.
\end{aligned}$$

To estimate these three terms, we shall use the following lemma.

Lemma 5. *Under the assumptions of Theorem* 1, *there exists a positive constant* $\delta = \delta(n, \deg P)$ *such that for any multi-index* α *and* $j \geqslant 0$,

(i) *$\beta - \delta - /\delta lk < 0$ and $\delta < \min\{k/2l, k/q'(k+l)\}$ and*

$$\left\| \tilde{T}_j^\alpha h \right\|_2 \leqslant C 2^{(\beta-\delta-\delta l/k)j} \|h\|_2; \tag{3.7}$$

(ii) *$\beta - \delta\sigma - \delta\sigma l/k < 0$ and $\delta < \min\{1/2, k/2l, k/q'\sigma(k+l)\}$ and*

$$\left\| \tilde{T}_j^\alpha h \right\|_p \leqslant C 2^{(\beta-\delta\sigma-\delta\sigma l/k)j} \|h\|_p, \tag{3.8}$$

where $1 < p < \infty$ $(p \neq 2)$, $\sigma = 1/2 - |1/p - 1/2|$.

Here C is independent of the coefficients of $P(x, y)$.

Proof. Let $b(r) = r^{|\alpha|-m}$ and $\bar{\Omega}(x) = \Omega(x)(x/|x|)^\alpha$. It is easy to see that $\bar{\Omega}(x)$ is homogeneous of degree zero and belongs to $L^q(S^{n-1})$. Note that

$$2^{j\beta} \int\limits_{2^{j-1} \leqslant |x-y| < 2^j} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} (x-y)^\alpha h(y)\, dy$$
$$= 2^{j\beta} \int\limits_{2^{j-1} \leqslant |x-y| < 2^j} e^{iP(x,y)} \frac{\bar{\Omega}(x-y)}{|x-y|^n} b\big(|x-y|\big) h(y)\, dy.$$

Checking the argument of Ding in [9, pp. 73–78], we can find that there exists a positive constant $\delta = \delta(n, \deg P)$ such that

(i) $\beta - \delta - \delta l/k < 0$ and $\delta < \min\{k/2l, k/q'(k+l)\}$ and

$$\left\| 2^{j\beta} \int\limits_{2^j \leqslant |x-y| < 2^{j+1}} e^{iP(x,y)} \frac{\bar{\Omega}(x-y)}{|x-y|^n} b\big(|x-y|\big) h(y)\, dy \right\|_2$$
$$\leqslant C 2^{(\beta-\delta-\delta l/k+|\alpha|-m)j} \|h\|_2;$$

(ii) $\beta - \delta\sigma - \delta\sigma l/k < 0$ and $\delta < \min\{1/2, k/2l, k/q'\sigma(k+l)\}$ and

$$\left\| 2^{j\beta} \int\limits_{2^j \leqslant |x-y| < 2^{j+1}} e^{iP(x,y)} \frac{\bar{\Omega}(x-y)}{|x-y|^n} b\big(|x-y|\big) h(y)\, dy \right\|_p$$
$$\leqslant C 2^{(\beta-\delta\sigma-\delta\sigma l/k+|\alpha|-m)j} \|h\|_p,$$

where $1 < p < \infty$ $(p \neq 2)$, $\sigma = 1/2 - |1/p - 1/2|$.

Here C is independent of the coefficients of $P(x, y)$. This leads to the conclusion of Lemma 5. □

We now return to the proof of Theorem 1. Let α be a multi-index such that $|\alpha| \leqslant m$, a straightforward computation (see [8, p. 452]) yields that

$$D^\alpha A_{\varphi_d}(y) = \sum_{\alpha=\mu+\nu} C_{\mu,\nu} R_{m-|\mu|}\left(D^\mu A_{Q_d}; y, x_0\right) D^\nu \varphi_d(y). \tag{3.9}$$

Recall that $\operatorname{supp}\varphi_d \subset 6\sqrt{n}\,Q_d$, we can get by Lemmas 1–3 that if $|\alpha| < m$,

$$\begin{aligned}
\left|D^\alpha A_{\varphi_d}(y)\right| &\leqslant \sum_{\alpha=\mu+\nu} C_{\mu,\nu}|y-x_0|^{m-|\mu|}|\bar{Q}_d|^{-|\nu|/n} \\
&\quad\times \sum_{|\eta|=m-|\mu|}\left(\frac{1}{|Q_y^{x_0}|}\int_{Q_y^{x_0}}\left|D^\eta(D^\mu A_{Q_d})(z)\right|^s dz\right)^{1/s} \\
&\leqslant C|Q_d|^{(m-|\alpha|)/n}\sum_{|\gamma|=m}\left(\frac{1}{|Q_y^{x_0}|}\int_{Q_y^{x_0}}\left|D^\gamma A_{Q_d}(z)\right|^s dz\right)^{1/s} \\
&\leqslant C\sum_{|\gamma|=m}\left(\frac{1}{|Q_y^{x_0}|}\int_{Q_y^{x_0}}\left|D^\gamma A(z)-m_{\bar{Q}_d}(D^\gamma A)\right|^s dz\right)^{1/s} \\
&\leqslant C|\bar{Q}_d|^{\beta/n}\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta} \leqslant C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta},
\end{aligned}$$

where $n < s < \infty$.

If $|\gamma| = m$, by (3.9) and Lemmas 1–3, we have

$$\begin{aligned}
\left|D^\alpha A_{\varphi_d}(y)\right| &\leqslant \sum_{\alpha=\mu+\nu,\,|\mu|<m} C_{\mu,\nu}\left|R_{m-|\mu|}(D^\mu A_{Q_d};y,x_0)D^\nu\varphi_d(y)\right| \\
&\quad+\sum_{|\alpha|=m}\left|\left(D^\alpha A(y)-m_{\bar{Q}_d}(D^\alpha A)\right)\varphi_d(y)\right| \\
&\leqslant C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}+\sum_{|\alpha|=m}\left|D^\alpha A(y)-m_{\bar{Q}_d}(D^\alpha A)\right| \\
&\leqslant C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta} \\
&\quad+\sum_{|\alpha|=m}\frac{1}{|\bar{Q}_d|^{1-\beta/n}}\int_{\bar{Q}_d}\sup_{y\in 6\sqrt{n}\,Q_d}\frac{|D^\alpha A(y)-D^\alpha A(x)|}{|y-x|^\beta}\,dx \\
&\leqslant C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}.
\end{aligned}$$

So, by Lemma 5 we obtain that

$$\|G\|_p \leqslant \|A_{\varphi_d}\|_\infty\left\|\tilde{T}_j^0 f_k\right\|_p \leqslant C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}2^{-j\theta}\|f_d\|_p, \tag{3.10}$$

where $\theta = \delta + \delta l/k - \beta > 0$ for $p = 2$, and $\theta = \delta(1/2 - |1/p - 1/2|) + \delta(1/2 - |1/p - 1/2|)l/k - \beta > 0$ for $1 < p < \infty$ $(p \neq 2)$. Similarly,

$$\|H\|_p \leqslant C\sum_{0<|\alpha|<m}\left\|\tilde{T}_j^\alpha(D^\alpha A_{\varphi_d}f_d)\right\|_p \leqslant C\sum_{0<|\alpha|<m}2^{-j\theta}\|D^\alpha A_{\varphi_d}\|_\infty\|f_d\|_p$$

$$\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{-j\theta} \|f_d\|_p$$

and

$$\|J\|_p \leqslant C \sum_{|\alpha|=m} \left\|\tilde{T}_j^\alpha (D^\alpha A_{\varphi_d} f_d)\right\|_p \leqslant C \sum_{|\alpha|=m} 2^{-j\theta} \|D^\alpha A_{\varphi_d}\|_\infty \|f_d\|_p$$
$$\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{-j\theta} \|f_d\|_p,$$

where θ is the same as in (3.10). This completes the proof of Theorem 1. □

4. Proof of Theorem 2

Let us start with a preliminary lemma.

Lemma 6 (cf. [12]). *Set*

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} \left|\Omega(x-y) f(y)\right| dy.$$

For $\Omega \in L^q(S^{n-1})$ $(1 < q < \infty)$, if p, q, and $\omega(x)$ satisfy one of the following conditions, then M_Ω is bounded on $L^p_\omega(\mathbb{R}^n)$, that is, $\|M_\Omega f\|_{p,\omega} \leqslant C\|f\|_{p,\omega}$.

(i) $q' \leqslant p < \infty$, $p \neq 1$, *and* $\omega \in A_{p/q'}$;
(ii) $1 < p \leqslant q$, $p \neq \infty$, *and* $\omega \in A_{p'/q'}$;
(iii) $1 < p < \infty$ *and* $\omega^{q'} \in A_p$.

Proof of Theorem 2. Similarly to the proof of Theorem 1, by dilation invariance, we may assume that $\sum_{|\xi|=k,\, |\eta|=l} |a_{\xi\eta}| = 1$. Write

$$T_A f(x) = T_A^0 f(x) + T_A^\infty f(x),$$

where $T_A^0 f(x)$ and $T_A^\infty f(x)$ are the same as in (3.1). From the proof of Theorem 1, we know that

$$\left|T_A^0 f(x)\right| \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} M_\Omega f(x).$$

By Lemma 6, under the assumptions of Theorem 2, we have

$$\left\|T_A^0 f\right\|_{p,\omega} \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|M_\Omega f\|_{p,\omega} \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_{p,\omega}, \tag{4.1}$$

where C is independent of the coefficients of $P(x, y)$.

It remains to prove that

$$\left\|T_A^\infty f\right\|_{p,\omega} \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_{p,\omega}. \tag{4.2}$$

Since

$$T_A^\infty f(x) = \sum_{j=0}^{\infty} \int\limits_{2^j \leqslant |x-y| < 2^{j+1}} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A; x, y) f(y)\, dy$$
$$= \sum_{j=0}^{\infty} T_A^j f(x),$$

we only need to prove that under the assumptions of Theorem 2, for each $j \in \mathbb{N}$, there exist positive constants ρ and C such that

$$\left\| T_A^j f \right\|_{p,\omega} \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{-\rho j} \|f\|_{p,\omega}. \tag{4.3}$$

In order to prove (4.3), we will use the following lemma.

Lemma 7. *Suppose that* $0 < \sigma \leqslant 1$, $0 < \beta < \sigma \min\{(l+k)/2k, (l+k)/2l\}$ *and* $q > 1/(1-\beta)$. *Then there exists* $\delta > 0$ *such that*

(i) $\beta - \delta\sigma - l\delta\sigma/k < 0$;
(ii) $\delta < \min\{k/2l, k/q'\sigma(l+k), 1/2\}$;

and

$$\left\| T_A^j f \right\|_2 \leqslant C 2^{(\beta-\delta-l\delta/k)j} \|f\|_2. \tag{4.4}$$

Proof. Notice that $\beta - \delta\sigma - l\delta\sigma/k < 0$ is equivalent to $\beta k/\sigma(l+k) < \delta$, we only need to prove

$$\frac{\beta k}{\sigma(l+k)} < \min\left\{\frac{1}{2}, \frac{k}{2l}, \frac{k}{q'\sigma(l+k)}\right\}.$$

From $0 < \beta < \sigma \min\{(l+k)/2k, (l+k)/2l\}$, we get that $\beta < \sigma(l+k)/2k$ and $\beta < \sigma(l+k)/2l$. Thus

$$\frac{\beta k}{\sigma(l+k)} < \frac{1}{2} \quad \text{and} \quad \frac{\beta k}{\sigma(l+k)} < \frac{k}{2l}.$$

On the other hand, we have $\beta < 1/q'$ from $q > 1/(1-\beta)$. Consequently, $\beta k/\sigma(l+k) < k/q'\sigma(l+k)$. Hence the above inequality holds and this proves (i) and (ii). And (4.4) can be obtained from the proof of (3.4). This proves Lemma 7. □

Now we return to the proof of Theorem 2. Applying Lemma 4, we have

$$\left|T_A^j f(x)\right| \leqslant \int\limits_{2^j \leqslant |x-y| < 2^{j+1}} \frac{|\Omega(x-y)|}{|x-y|^{n+m}} \left|R_{m+1}(A; x, y)\right| \left|f(y)\right| dy$$
$$\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{j\beta} 2^{-jn} \int\limits_{|x-y| < 2^j} \left|\Omega(x-y)\right| \left|f(y)\right| dy$$

$$\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} 2^{j\beta} M_\Omega f(x). \tag{4.5}$$

Next we will prove (4.2) under the assumptions (i), (ii), and (iii) of Theorem 2, respectively.

(i) $2 < p < \infty$, $q' \leqslant p$, $\omega(x) \in A_{p/q'}$ and $\varepsilon \in (0,1)$ such that $\omega(x)^{1+\varepsilon} \in A_{p/q'}$.

At first, we claim that $q > 1/(1-\beta)$ under the above assumption. In fact, since $\min\{(l+k)/2k, (l+k)/2l\} \leqslant 1$ and $\varepsilon < 1$, thus $\beta < \sigma = 2\varepsilon/p(1+\varepsilon) < 1/p$. This implies $1 < 1/(1-\beta) < p'$. It is automatically deduced from $q' \leqslant p$ that $q > 1/(1-\beta)$.

Let $p_1 = (1+\varepsilon)(p-2)+2$. Then $2 < p < p_1 < (1+\varepsilon)p$ and $q' \leqslant p < p_1$. By (b) of the properties of A_p weights, we know that $\omega(x)^{1+\varepsilon} \in A_{p/q'} \subset A_{p_1/q'}$. Invoking Lemma 6(i) and (4.4), we get that

$$\begin{aligned}\|T_A^j f\|_{p_1,\omega^{1+\varepsilon}} &\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} 2^{j\beta} \|M_\Omega f\|_{p_1,\omega^{1+\varepsilon}} \\ &\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} 2^{j\beta} \|f\|_{p_1,\omega^{1+\varepsilon}}.\end{aligned} \tag{4.6}$$

Denote $\theta_1 = p_1/(1+\varepsilon)p$, then $0 < \theta_1 < 1$ and $1/p = (1-\theta_1)/2 + \theta_1/p_1$. Applying the interpolation theorem with change of measure of Stein–Weiss [1] to (4.4) and (4.6), we can get that

$$\begin{aligned}\|T_A^j f\|_{p,\omega} &\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} 2^{j\beta\theta_1 + (1-\theta_1)(\beta-\delta-l\delta/k)j} \|f\|_{p,\omega} \\ &= \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} 2^{[\beta-(1-\theta_1)\delta-(1-\theta_1)\delta l/k]j} \|f\|_{p,\omega}.\end{aligned}$$

Observe that $1-\theta_1 = 2\varepsilon/p(1+\varepsilon) = \sigma$, it is obvious from Lemma 7 that

$$\beta - (1-\theta_1)\delta - (1-\theta_1)\delta l/k = \beta - \sigma\delta - \sigma\delta l/k < 0.$$

Denote $\rho = \sigma\delta - \delta\sigma l/k - \beta$. Then $\rho > 0$ and

$$\|T_A^j f\|_{p,\omega} \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} 2^{-\rho j} \|f\|_{p,\omega}.$$

This proves (4.2) under the assumption (i).

(ii) $1 < p < 2$, $p \leqslant q$, $\omega(x)^{-1/(p-1)} \in A_{p'/q'}$. For μ satisfying that $(\omega^{-1/(p-1)})^{1+\mu} \in A_{p'/q'}$, choose ε such that

$$0 < \varepsilon < \min\left\{\frac{p-1}{2}, \frac{\mu(p-1)}{(1+\mu)(2-p)}\right\} \quad \text{and} \quad (\omega^{-1/(p-1)})^{(1+\mu)(1+\varepsilon)} \in A_{p'/q'}.$$

Similarly to the proof of (i), we start with the claim: $q > 1/(1-\beta)$ under the above assumption. In fact, since $0 < \varepsilon < (p-1)/2$, we have

$$\beta < \sigma = 2\varepsilon/p(1+\varepsilon) < (p-1)/p(1+\varepsilon) < 1/p',$$

that is, $1 < 1/(1-\beta) < p$. Therefore, $q > 1/(1-\beta)$ from $p \leqslant q$.

By Lemma 7, there exists $\delta > 0$ such that

$$\beta - \delta\sigma - l\delta\sigma/k < 0 \quad \text{and} \quad \delta < \min\left\{\frac{1}{2}, \frac{k}{2l}, \frac{k}{q'\sigma(l+k)}\right\},$$

and (4.4) holds.

Now let $p_2 = 2 - (1+\varepsilon)(2-p)$; then $1+\varepsilon < p-\varepsilon < p_2 < p$. Consequently,

$$p_2 < p \leqslant q \quad \text{and} \quad 1 + (p-1)/(1-\mu) < p_2 < p.$$

This implies $0 < (p-1)/(1+\mu)(p_2-1) < 1$.

By (a) and (b) of the properties of A_p weights and $(\omega^{-1/(p-1)})^{(1+\mu)(1+\varepsilon)} \in A_{p'/q'}$, we know that

$$(\omega^{-1/(p_2-1)})^{1+\varepsilon} = \left\{(\omega^{-1/(p-1)})^{(1+\mu)(1+\varepsilon)}\right\}^{(p-1)/(1+\mu)(p_2-1)} \in A_{p'/q'} \subset A_{p_2'/q'}.$$

Now Lemma 6(ii) and (4.5) state that

$$\begin{aligned}\left\|T_A^j f\right\|_{p_2,\omega^{1+\varepsilon}} &\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{j\beta} \|M_\Omega f\|_{p_2,\omega^{1+\varepsilon}} \\ &\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{j\beta} \|f\|_{p_2,\omega^{1+\varepsilon}}. \end{aligned} \tag{4.7}$$

Set $\theta_2 = p_2/(1+\varepsilon)p$; then $0 < \theta_2 < 1$ and $1/p = (1-\theta_2)/2 + \theta_2/p_2$. Observe that $1^{p(1-\theta_2)/2}\omega^{(1+\varepsilon)p\theta_2/2} = \omega$. Interpolating with change of measure between (4.4) and (4.7), we have

$$\begin{aligned}\left\|T_A^j f\right\|_{p,\omega} &\leqslant C 2^{\beta j\theta_2 + (1-\theta_2)(\beta-\delta-\delta l/k)j} \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_{p,\omega} \\ &= C 2^{j[\beta-(1-\theta_2)\delta(l+k)/k]} \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_{p,\omega}.\end{aligned}$$

Since $1-\theta_2 = 2\varepsilon/p(1+\varepsilon) = \sigma$, we have

$$\rho := (1-\theta_2)\delta(l+k)/k - \beta = \sigma\delta + \sigma\delta l/k - \beta > 0.$$

Thus

$$\left\|T_A^j f\right\|_{p,\omega} \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{-\rho j} \|f\|_{p,\omega},$$

which implies (4.2) under the assumption (ii).

(iii) $p = 2$, $q > 1/(1-\beta)$, $\omega^{q'} \in A_2$ and $\varepsilon > 0$ such that $(\omega^{q'})^{1+\varepsilon} \in A_2$.

Similarly to the argument as those in the proof of (i) and (ii), we can deduce that there exists $\delta > 0$ such that

$$\beta - \delta\sigma - l\delta\sigma/k < 0 \quad \text{and} \quad \delta < \min\left\{\frac{1}{2}, \frac{k}{2l}, \frac{k}{q'\sigma(l+k)}\right\}$$

and (4.4) holds.

Also, $(\omega^{q'})^{1+\varepsilon} \in A_2$ and (c) of the properties of A_p weights and (4.2) show that

$$\begin{aligned}\left\|T_A^j f\right\|_{2,\omega^{1+\varepsilon}} &\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{j\beta} \|M_\Omega f\|_{p,\omega^{1+\varepsilon}} \\ &\leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{j\beta} \|f\|_{2,\omega^{1+\varepsilon}}. \end{aligned} \tag{4.8}$$

Denote $\theta_3 = 1/(1+\varepsilon)$, then $0 < \theta_3 < 1$ and $1/2 = (1-\theta_3)/2 + \theta_3/2$. Then

$$1^{2(1-\theta_3)/2} \omega^{(1+\varepsilon)2\theta_3/2} = \omega.$$

Invoking the interpolation theorem with change of measure of Stein–Weiss again, we have

$$\left\|T_A^j f\right\|_{2,\omega} \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{j[\beta-(1-\theta_3)(\delta+\delta l/k)]} \|f\|_{2,\omega}.$$

Since $1-\theta_3 = \varepsilon/(1+\varepsilon) = \sigma$, thus $\rho := (1-\theta_3)(\delta + \delta l/k) - \beta = \delta\sigma - l\delta\sigma/k - \beta > 0$. This shows that

$$\left\|T_A^j f\right\|_{2,\omega} \leqslant C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} 2^{-j\rho} \|f\|_{2,\omega},$$

which proves (4.2) under the assumption (iii). Theorem 2 is proved. □

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